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Mathematical Analysis of the Stochastic Dynamics of a Spinning Spherical Satellite*

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The vector differential equation describing the motion of a spinning spherical satellite is here studied by assuming that the aerodynamical forces have random nature. The resulting evolution equation is a random differential equation with stochastic process coefficients which is solved by using a perturbation procedure and by following known methods of stochastic systems analysis. The solution process is therefore found in an approximated analytical form, which allows the determination of some statistical properties of the system. © 1985 Academic Press, Inc.

1. INTRODUCTION

The problem of the dynamics of a spinning Earth-satellite is a classical one which has been studied in deterministic terms by many authors, see, for instance, the book of Leimanis [1] and Refs. [2–4] with their related bibliographies. Generally in these works the influence of aerodynamical actions on the satellite is neglected or oversimplified by assuming that the aerodynamical force is constant.

However, as was discussed, for instance, in the review paper [5], a more accurate analysis of the dynamical problem is needed when the theoretical approach is requested to supply valuable predictions concerning flight attitude, life-time and changes in satellite orbits, which knowledge is of primary importance for the applications.

In a more accurate mathematical formulation of the physical problem, the following two considerations cannot be neglected. First, the aerodynamical actions applied to the space vehicle in upper-atmosphere flow conditions (200–600 km altitude) must be defined by considering the force and torque coefficients as functions of the dynamical state of the system itself, so that the dynamics and aerodynamics of the satellite cannot be studied separately. Second, it must be observed that during the satellite's flight the physical

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conditions of the upper atmosphere are subjected to irregular variations which are mainly due (see Ref. [6]) to fluctuations of the air density.

Consequently, in this paper the dynamics of a spinning spherical earth satellite is studied by considering the presence of unsteady aerodynamical forces and their random nature. The mathematical problem is treated by writing the evolution equation of the system in the form of a vector random differential equation with stochastic process coefficients and deterministic initial conditions. This equation, which is deduced in Section 2, is nonlinear, owing to the presence of both the aerodynamical and the Newtonian field forces. The study of the evolution equation is dealt with in Section 3 by applying a perturbation technique [7] and is based on the mathematical methods of solution of random differential equations [8–12]. Then, approximated expressions for the first- and second-order moments of the solution process are deduced in Section 4, where a method for computing the first probability density is also developed, by following the methods proposed in Ref. [10] and under the assumption of suitable conditions for the stochastic process coefficients of the considered differential equation.

2. Mathematical Description of the System and Derivation of the Motion Equation

In the present section the stochastic differential equation, which is studied in this paper, is derived by recalling the physical-mathematical axioms of the considered problem.

Referring to Fig. 1 consider a spherical body of radius R and mass m moving in a rarefied monoatomic gas with density ρ and call x_1, x_2, x_3 the coordinates of the center of mass G at the time $t \in I = [0, T]$ and θ_1, θ_2, θ_3 the Eulerian angles. Accordingly, the following assumptions are made:

A1. The dynamical state of the sphere is defined by the dimensionless vector

$$\mathbf{y} = \mathbf{y}(t, \omega)\colon \Omega \times I \to D_y \subset \mathbb{R}^{12}; \qquad \omega \in (\Omega, F, \mu), \tag{1}$$

$$\begin{aligned}\mathbf{y} = \{ & y_1 = x_1/r_0,\, y_2 = x_2/r_0,\, y_3 = x_3/r_0,\, y_4 = \theta_1,\, y_5 = \theta_2,\, y_6 = \theta_3, \\ & y_7 = \dot{x}_1/V_0,\, y_8 = \dot{x}_2/V_0,\, y_9 = \dot{x}_3/V_0,\, y_{10} = Rq_1/V_0, \\ & y_{11} = Rq_2/V_0,\, y_{12} = Rq_3/V_0\}\end{aligned} \tag{1$'$}$$

with $y_4 \in (0, \pi)$, $y_5 \in [0, 2\pi)$, $y_6 \in [0, 2\pi)$, and where r_0, V_0 are, respectively, the distance from O and the speed of the center of mass G at the time $t = 0$; q_1, q_2, q_3 are the components of the instantaneous rotation $\vec{q}$ with respect to $O(x_1, x_2, x_3)$.

A2. The sphere is moving in a gravitational field

$$\vec{F}_g = -\gamma m_e \frac{m}{r^3} \overrightarrow{OG} \tag{2}$$

where m_e is the mass of the Earth and γ is the gravitational constant.

A3. The aerodynamical force and torque acting on the sphere are given, respectively, by

$$\vec{F}_a = \tfrac{1}{2}\rho V_G^2 \pi R^2 (C_1 \vec{\imath}_1 + C_2 \vec{\imath}_2 + C_3 \vec{\imath}_3) \tag{3}$$

$$\vec{M}_1 = \tfrac{1}{2}\rho V_G^2 \pi R^3 (C_{M1} \vec{\imath}_1 + C_{M2} \vec{\imath}_2 + C_{M3} \vec{\imath}_3) \tag{4}$$

where C_k and C_{Mk}, $k = 1, 2, 3$, are the aerodynamical coefficients of force and torque, assumed as known analytical functions of the components $y_7,..., y_{12}$ of the state vector $\mathbf{y}$, according to known hypersonic mathematical models of the gas–surface interaction [13].

A4. A probabilistic model for the air mass density ρ is assumed in the following general form:

$$\rho(\mathbf{y}, \omega, t) = \rho_d(\mathbf{y})[1 + X(t, \omega)] \tag{5}$$

where $X(t, \omega)$ is a bounded, zero-mean stochastic process defined on $I \times \Omega$, with $\omega \in (\Omega, F, \mu)$, a complete probability space; and $\rho_d(\mathbf{y})$ is a deterministic model of the upper-atmosphere density as a function of altitude. By following [6, 14], it is assumed that $\rho_d(\mathbf{y})$ is given by

$$\rho_d(\mathbf{y}) = \rho_0 \exp \left\{ -\frac{1}{H} [r_0(y_1^2 + y_2^2 + y_3^2)^{1/2} - R_e] \right\} \tag{6}$$

where the height scale H and the constant ρ_0 are to be deduced from experimental data (see, for instance, Ref. [15]), and R_e is the radius of the Earth.

Suitable explicit expressions of the stochastic process $X(t, \omega)$ appearing in Eq. (6) should be considering the available data about the fluctuations of the air density during the satellite flight. Valuable detailed results on this argument are supplied, for instance, in the quoted Refs. [6, 14] and related bibliographies. However, this matter, being mainly based on the analysis of experimental data, is not dealt with in the present paper, which is concerned with the theoretical analysis of the stochastic behaviour of the system.

According to the above assumptions the classical equations of the dynamics of the system can be written in the following vector form:

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) + \varepsilon \mathbf{f}(\mathbf{y})[1 + X(t, \omega)], \qquad \mathbf{y}(0) = \mathbf{y}_0 \tag{7}$$

with

$$\varepsilon = \frac{\rho_0 \pi R^3}{2m} \exp[-(r_0 - R_e)/H] \tag{8}$$

$$\mathbf{g} = \{g_i\}, \qquad \mathbf{f} = \{f_i\} \qquad i = 1,..., 12$$

$$g_1 = V_0 y_7 / r_0 \tag{9a}$$

$$g_2 = V_0 y_8 / r_0 \tag{9b}$$

$$g_3 = V_0 y_9 / r_0 \tag{9c}$$

$$g_4 = V_0(y_{10} \cos y_6 + y_{11} \sin y_6)/R \tag{9d}$$

$$g_5 = V_0(y_{10} \sin y_6 - y_{11} \cos y_6)/(R \sin y_4) \tag{9e}$$

$$g_6 = V_0\{y_{12} - (y_{10} \sin y_6 - y_{11} \cos y_6)/\tan y_4\}/R \tag{9f}$$

$$g_7 = -\gamma m_e y_1 / \{r_0 V_0 (y_1^2 + y_2^2 + y_3^2)^{3/2}\} \tag{9g}$$

$$g_8 = -\gamma m_e y_2 / \{r_0 V_0 (y_1^2 + y_2^2 + y_3^2)^{3/2}\} \tag{9h}$$

$$g_9 = -\gamma m_e y_3 / \{r_0 V_0 (y_1^2 + y_2^2 + y_3^2)^{3/2}\} \tag{9i}$$

$$g_{10} = g_{11} = g_{12} = 0 \tag{9j}$$

$$f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0 \tag{10a}$$

$$f_7 = V_0 \exp\{r_0[1 - (y_1^2 + y_2^2 + y_3^2)^{1/2}]/H\}(y_7^2 + y_8^2 + y_9^2)\, C_1(\mathbf{y})/R \tag{10b}$$

$$f_8 = V_0 \exp\{r_0[1 - (y_1^2 + y_2^2 + y_3^2)^{1/2}]/H\}(y_7^2 + y_8^2 + y_9^2)\, C_2(\mathbf{y})/R \tag{10c}$$

$$f_9 = V_0 \exp\{r_0[1 - (y_1^2 + y_2^2 + y_3^2)^{1/2}]/H\}(y_7^2 + y_8^2 + y_9^2)\, C_3(\mathbf{y})/R \tag{10d}$$

$$f_{10} = 5V_0 \exp\{r_0[1 - (y_1^2 + y_2^2 + y_3^2)^{1/2}]/H\}(y_7^2 + y_8^2 + y_9^2)\, C_{M1}(\mathbf{y})/(2R) \tag{10e}$$

$$f_{11} = 5V_0 \exp\{r_0[1 - (y_1^2 + y_2^2 + y_3^2)^{1/2}]/H\}(y_7^2 + y_8^2 + y_9^2)\, C_{M2}(\mathbf{y})/(2R) \tag{10f}$$

$$f_{12} = 5V_0 \exp\{r_0[1 - (y_1^2 + y_2^2 + y_3^2)^{1/2}]/H\}(y_7^2 + y_8^2 + y_9^2)\, C_{M3}(\mathbf{y})/(2R). \tag{10g}$$

As a consequence of the axiom A4, Eq. (7) is a nonlinear random differential equation with stochastic process coefficients and deterministic initial conditions. Moreover, it may be easily verified that in upper-atmosphere flight conditions the dimensionless quantity ε defined by Eq. (8) is of a smaller order compared with unity.

Remark. The particular cases $y_4 = 0$ or $y_4 = \pi$, which are excluded in the above analysis, can be treated separately by recalling that, as known [16], in

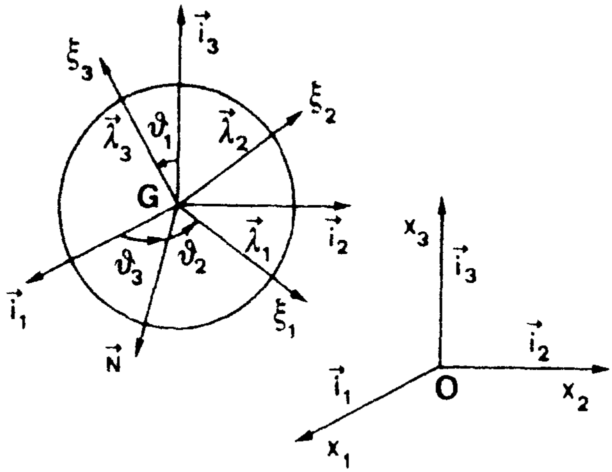

FIG. 1. Geometry of the system.

these cases the Eulerian angles $\theta_2 = y_5$ and $\theta_3 = y_6$ are undetermined, whereas the actual position of the sphere with respect to G is uniquely determined by the condition $y_4 = 0$ (or $y_4 = \pi$) and by the value of the sum $y_5 + y_6$.

3. Analysis

In this section an approximated explicit solution of Eq. (7) is derived by following the methods of perturbation theory [7] suitably extended to the study of random differential equations [8–12]. Taking into account the presence in Eq. (7) of the small parameter ε, its solution for $t \in [0, T]$ can be searched in the approximated form:

$$\mathbf{y} \cong \mathbf{y}^*(t, \omega) = \mathbf{y}^{(0)}(t; \varepsilon = 0) + \varepsilon \mathbf{y}^{(1)}(t, \omega) \tag{11}$$

$$\mathbf{y}^{(0)}(t = 0) = \mathbf{y}_0, \qquad \mathbf{y}^{(1)}(t = 0) = \mathbf{0}. \tag{12}$$

The terms $\mathbf{y}^{(0)}$, $\mathbf{y}^{(1)}$ of the truncated expansion (11) can be determined in analytical form since, as follows from Eqs. (9), (10) and the axiom A3, $f(\mathbf{y})$ and $g(\mathbf{y})$ are differentiable functions of y_i with bounded derivatives, whose components can be expanded in powers of ε:

$$f_i = f_{i, \varepsilon = 0} + \varepsilon \sum_{j=1}^{12} \left(\frac{\partial f_i}{\partial y_j} \frac{\partial y_j}{\partial \varepsilon} \right)_{\varepsilon = 0} + O(\varepsilon^2) \tag{13}$$

$$g_i = g_{i, \varepsilon = 0} + \varepsilon \sum_{j=1}^{12} \left(\frac{\partial g_i}{\partial y_j} \frac{\partial y_j}{\partial \varepsilon} \right)_{\varepsilon = 0} + O(\varepsilon^2). \tag{14}$$

Inserting Eqs. (11)–(14) into Eq. (7) and equating the terms of like powers in ε, the two following differential equations for $\mathbf{y}^{(0)}$ and $\mathbf{y}^{(1)}$ are obtained:

$$\dot{\mathbf{y}}^{(0)} = \mathbf{g}(\mathbf{y}^{(0)}), \qquad \mathbf{y}^{(0)}(t=0) = \mathbf{y}_o \tag{15}$$

$$\dot{\mathbf{y}}^{(1)} = \tilde{\mathbf{g}}(\mathbf{y}^{(0)}, \mathbf{y}^{(1)}) + \{1 + X(t, \omega)\}\, \mathbf{f}(\mathbf{y}^{(0)}), \qquad \mathbf{y}^{(1)}(t=0) = \mathbf{0} \tag{16}$$

with

$$\tilde{\mathbf{g}} = \{\tilde{g}_i\}; \qquad \tilde{g}_i(\mathbf{y}^{(0)}, \mathbf{y}^{(1)}) = \sum_{j=1}^{12} \left(\frac{\partial g_i}{\partial y_j} \frac{\partial y_j}{\partial \varepsilon}\right)_{\varepsilon=0}. \tag{16'}$$

Equation (15) is a nonlinear deterministic differential equation whose solution gives the "unperturbed" inertial motion of the spinning sphere. Insertion of this solution $\mathbf{y}^{(0)}$ into Eq. (16) leads to an initial value problem for the term $\mathbf{y}^{(1)}$ which, as further shown, is described by a linear differential equation with random inhomogeneous part. Higher-order terms of the expansion (1) could be considered in the development of the above procedure; nevertheless, they are neglected in the present work since in the actual physical conditions of the satellite's flight the parameter ε must be considered of the order of 10^{-9}.

Let us now deal with the solutions of Eqs. (15) and (16).

Unperturbed Motion

Equation (15) is equivalent to two decoupled systems of differential equations describing the motion of the center of mass G and the motion relative to G. As known, these represent, respectively, the Kepler and Poinsot problems, whose solutions may be found in the classical literature (see, for instance, [17, 1]). In particular, if we introduce the dimensionless vector variable

$$\begin{gathered}\mathbf{u} = \{u_1 = y_1, u_2 = y_2, u_3 = y_3, u_4 = y_7, u_5 = y_8, u_6 = y_9\} \in \mathbb{R}^6 \\ \mathbf{u}(t=0) = \mathbf{u}_0; \qquad \mathbf{u}^* \cong \mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)}\end{gathered} \tag{17}$$

defining the motion of G, then the Kepler's solution may be written for circular orbits and according to the initial conditions

$$\mathbf{u}_0 = \{u_{1,0} = 1, u_{2,0} = u_{3,0} = u_{4,0} = 0, u_{5,0} = r_0 \nu / V_0, u_{6,0} = 0\} \tag{18}$$

in the following explicit form:

$$\begin{aligned} u_1^{(0)}(t) &= \cos \nu t, & u_4^{(0)}(t) &= -u_{5,0} \sin \nu t \\ u_2^{(0)}(t) &= \sin \nu t, & u_5^{(0)}(t) &= u_{5,0} \cos \nu t \\ u_3^{(0)} &= 0, & u_6^{(0)} &= 0 \end{aligned} \tag{19}$$

where

$$\nu = (2|E|/m)^{3/2}/(\gamma m_e); \qquad E = -m^3\gamma^2 m_e^2/(2K_0^2) \tag{20}$$

K_0 being the satellite's angular momentum with respect to 0.

With regard to the motion relative to G, introducing the dimensionless vector

$$\begin{gathered} \mathbf{v} = \{v_1 = y_4, v_2 = y_5, v_3 = y_6, v_4 = y_{10}, v_5 = y_{11}, v_6 = y_{12}\} \in \mathbb{R}^6 \\ \mathbf{v}(t=0) = \mathbf{v}_0 = \{v_{n,0}\}, \qquad n = 1,..., 6; \qquad \mathbf{v}^* = \mathbf{v}^{(0)} + \varepsilon \mathbf{v}^{(1)} \end{gathered} \tag{21}$$

and assuming that for $t = 0$ the moving central axis of inertia ξ_3 (see Fig. 1) is coincident with the direction of $\vec{q}$, then the solution $\mathbf{v}^{(0)}$ of the Poinsot problem is

$$\begin{gathered} v_1^{(0)} = v_{1,0} = \cos^{-1}[v_{6,0}/(v_{4,0}^2 + v_{5,0}^2 + v_{6,0}^2)^{1/2}] \\ v_2^{(0)}(t) = \frac{V_0}{R}(v_{4,0}^2 + v_{5,0}^2 + v_{6,0}^2)^{1/2} t + v_{2,0} \\ v_3^{(0)} = v_{3,0} = \tan^{-1}(-v_{4,0}/v_{5,0}) \\ v_4^{(0)} = v_{4,0}; \qquad v_5^{(0)} = v_{5,0}; \qquad v_6^{(0)} = v_{6,0}. \end{gathered} \tag{22}$$

Perturbed Motion of the Center of Mass

Let us now deal with Eq. (16) by first considering the motion of the center of mass, which is described by the vector $\mathbf{u}$. After calculation of the derivatives in Eq. (16′), the following equation is obtained:

$$\dot{\mathbf{u}}^{(1)} + A(t)\,\mathbf{u}^{(1)} = \mathbf{Y}_u(t, X(t, \omega)); \qquad \mathbf{u}^{(1)}(0) = \mathbf{0} \tag{23}$$

where

$$\mathbf{Y}_u = \frac{V_0}{R} u_{5,0}^2 (1 + X(t, \omega))\{0, 0, 0, C_1(t), C_2(t), C_3(t)\} \tag{23′}$$

and the 6×6 matrix $A(t) = \{a_{ij}(t)\}$ has the following *non-null* elements:

$$\begin{gathered} a_{14} = a_{25} = a_{36} = -V_0/r_0 \\ a_{41}(t) = \gamma m_e(1 - 3\cos^2 \nu t)/(r_0^2 V_0) \\ a_{42}(t) = a_{51}(t) = -3\gamma m_e \cos \nu t \cdot \sin \nu t/(r_0^2 V_0) \\ a_{52}(t) = \gamma m_e(1 - 3\sin^2 \nu t)/(r_0^2 V_0) \\ a_{63} = \gamma m_e/(r_0^2 V_0). \end{gathered} \tag{24}$$

In Eq. (23′), the aerodynamical coefficients $C_k(t) = C_k(\mathbf{u}^{(0)}, \mathbf{v}^{(0)})$ are known functions of time, since they are defined in terms of the unperturbed solution $\mathbf{u}^{(0)}(t)$ and $\mathbf{v}^{(0)}(t)$.

Consequently Eq. (23) is a linear differential equation with time-dependent coefficients and random inhomogeneous part, whose statistical properties are known when the stochastic process $X(t, \omega)$ is defined. The solution of the above stochastic equation may be found by application of the Adomian decomposition method [9], which, in view of finding statistical measures of the solution process, gives remarkable advantages over the successive approximation methods used in the deterministic cases (see Chapter VIII of Ref. [9]). A further advantage is that the method solves nonlinear equations without linearization resulting in more physically realistic solutions.

According to Adomian's method, the solutions process $\mathbf{u}^{(1)}$ is given by

$$\mathbf{u}^{(1)}(t, \omega) = \sum_{l=0}^{\infty} (-1)^l (L^{-1}A)^{(l)} L^{-1} \mathbf{Y}_u(t, \omega) \tag{25}$$

where L is the linear differential operator and the superscript (l) means l-times application of the operator $L^{-1}A$ to the vector $L^{-1}\mathbf{Y}_u$; the convergence of the series (25) is assured [9] for $t \in [0, T]$ in the hypotheses stated in Section 2.

Perturbed Rotational Motion

The ε-order perturbed motion about the center of mass is described, from Eqs. (16), (16′), by

$$\dot{\mathbf{v}}^{(1)} = B\mathbf{v}^{(1)} + \mathbf{Y}_v(t, X(t, \omega)), \qquad \mathbf{v}^{(1)}(0) = \mathbf{0} \tag{26}$$

where

$$\mathbf{Y}_v = \frac{5V_0}{2R} u_{5,0}^2 (1 + X(t, \omega))\{0, 0, 0, C_{M1}(t), C_{M2}(t), C_{M3}(t)\} \tag{26′}$$

with $C_{Mk}(t) = C_{Mk}(\mathbf{u}^{(0)}(t), \mathbf{v}^{(0)}(t))$ in analogy with Eq. (23′) and $B = \{b_{ij}\}$ a 6×6 constant matrix whose *non-null* elements are the following functions of the initial conditions of motion:

$$\begin{aligned}
&b_{13}(\mathbf{v}_0) = -q_0' = -\frac{V_0}{R}(v_{4,0}^2 + v_{5,0}^2)^{1/2}, && b_{14}(\mathbf{v}_0) = -V_0^2 v_{5,0}/(R^2 q_0') \\
&b_{15}(\mathbf{v}_0) = V_0^2 v_{4,0}/(R^2 q_0'), && b_{21}(\mathbf{v}_0) = -V_0 q_0 v_{6,0}/(R q_0') \\
&b_{24}(\mathbf{v}_0) = -V_0^2 q_0 v_{4,0}/(R^2 q_0'^2), && b_{25}(\mathbf{v}_0) = V_0^2 q_0 v_{5,0}/(R^2 q_0'^2) \\
&b_{31}(\mathbf{v}_0) = q_0^2/q_0', && b_{34}(\mathbf{v}_0) = -V_0^3 v_{4,0} v_{6,0}/(R^3 q_0'^2) \\
&b_{35}(\mathbf{v}_0) = -V_0^3 v_{5,0} v_{6,0}/(R^3 q_0'^2), && b_{36}(\mathbf{v}_0) = V_0/R
\end{aligned} \tag{27}$$

$$q_0 = \frac{V_0}{R}(v_{4,0}^2 + v_{5,0}^2 + v_{6,0}^2)^{1/2}.$$

Owing to the properties of the matrix B the solution of Eq. (26) can be obtained by standard procedure as follows (see [8]):

$$\{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\} = \frac{5V_0}{2R} u_{5,0} \Phi(t) \int_0^t \Phi^{-1}(s)\, \mathbf{I}(s, \omega)\, ds \tag{28}$$

where

$$\Phi(t) = \begin{bmatrix} \cos q_0 t & 0 & b_{13} \sin(q_0 t)/q_0 \\ b_{21} \sin(q_0 t)/q_0 & 1 & b_{13} b_{21}(1 - \cos q_0 t)/q_0^2 \\ -q_0 \sin(q_0 t)/b_{13} & 0 & \cos q_0 t \end{bmatrix} \tag{29}$$

and $\mathbf{I} = \{I_1, I_2, I_3\}$, with

$$\begin{aligned} I_1(s, \omega) &= \int_0^s [b_{14} C_{M1}(\tau) + b_{15} C_{M2}(\tau)][1 + X(\tau, \omega)]\, d\tau \\ I_2(s, \omega) &= \int_0^s [b_{24} C_{M1}(\tau) + b_{25} C_{M2}(\tau)][1 + X(\tau, \omega)]\, d\tau \\ I_3(s, \omega) &= \int_0^s [b_{34} C_{M1}(\tau) + b_{35} C_{M2}(\tau) + b_{36} C_{M3}(\tau)][1 + X(\tau, \omega)]\, d\tau \end{aligned} \tag{30}$$

and moreover

$$\begin{aligned} v_4^{(1)}(t, \omega) &= \frac{5V_0}{2R} u_{5,0}^2 \int_0^t [1 + X(s, \omega)]\, C_{M1}(s)\, ds \\ v_5^{(1)}(t, \omega) &= \frac{5V_0}{2R} u_{5,0}^2 \int_0^t [1 + X(s, \omega)]\, C_{M2}(s)\, ds \\ v_6^{(1)}(t, \omega) &= \frac{5V_0}{2R} u_{5,0}^2 \int_0^t [1 + X(s, \omega)]\, C_{M3}(s)\, ds. \end{aligned} \tag{31}$$

The above m.s. integrals, as well as the solution (25) for $\mathbf{u}^{(1)}$, can be explicitly calculated when a suitable model of the stochastic process $X(t, \omega)$ is defined. Concluding, the approximated solition of the dynamical problem (7) has been obtained as follows:

—*motion of the center of mass*:

$$\mathbf{u}^*(t, \omega) = \mathbf{u}^{(0)}(t) + \varepsilon \mathbf{u}^{(1)}(t, \omega) \tag{32}$$

where $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ are given, respectively, by Eqs. (19), (25);

—*rotational motion*:

$$\mathbf{v}^*(t, \omega) = \mathbf{v}^{(0)}(t) + \varepsilon \mathbf{v}^{(1)}(t, \omega) \tag{33}$$

where $\mathbf{v}^{(0)}$ and $\mathbf{v}^{(1)}$ are supplied by Eqs. (22), (28), (31).

4. Statistical Measures and Conclusions

The explicit approximated solution process derived in the preceding section is now applied to the calculation of the most important statistical properties of the dynamical system.

Moments

The expectations of the state vectors $\mathbf{u}^*(t, \omega)$ and $\mathbf{v}^*(t, \omega)$ are obtained from their definitions and Eqs. (32), (33):

$$E\{\mathbf{u}^*\} = \mathbf{u}^{(0)}(t) + \varepsilon \sum_{l=0}^{\infty} (-1)^l (L^{-1}A)^{(l)} L^{-1} E\{\mathbf{Y}_u\} \tag{34}$$

$$E\{\mathbf{v}^*\} = \mathbf{v}^{(0)}(t) + \varepsilon E\{\mathbf{v}^{(1)}\}. \tag{35}$$

Recalling that the stochastic process $X(t, \omega)$ has null mean value (axiom A4), it follows from the results of the analysis developed in Section 3 that the expectations of $\mathbf{Y}_u$ and $\mathbf{v}^{(1)}$ appearing in the above equations are

$$E\{\mathbf{Y}_u\} = \frac{V_0}{R} u_{5,0}^2 \{0, 0, 0, C_1(t), C_2(t), C_3(t)\} \tag{36}$$

$$\{E\{v_1^{(1)}\}, E\{v_2^{(1)}\}, E\{v_3^{(1)}\}\} = \frac{5V_0}{2R} u_{5,0}^2 \Phi(t) \int_0^t \Phi^{-1}(s) E\{\mathbf{I}\} \, ds \tag{37}$$

with

$$E\{I_1\} = \int_0^s [b_{14} C_{M1}(\tau) + b_{15} C_{M2}(\tau)] \, d\tau \tag{38a}$$

$$E\{I_2\} = \int_0^s [b_{24} C_{M1}(\tau) + b_{25} C_{M2}(\tau)] \, d\tau \tag{38b}$$

$$E\{I_3\} = \int_0^s [b_{34} C_{M1}(\tau) + b_{35} C_{M2}(\tau) + b_{36} C_{M3}(\tau)] \, d\tau \tag{38c}$$

and moreover

$$E\{v_4^{(1)}\} = \frac{5V_0}{2R} u_{5,0}^2 \int_0^t C_{M1}(s) \, ds \tag{38d}$$

$$E\{v_5^{(1)}\} = \frac{5V_0}{2R} u_{5,0}^2 \int_0^t C_{M2}(s) \, ds \tag{38e}$$

$$E\{v_6^{(1)}\} = \frac{5V_0}{2R} u_{5,0}^2 \int_0^t C_{M3}(s) \, ds \tag{38f}$$

where the constants b_{ij} are defined by Eq. (27). The mean solution of the dynamical problem is therefore obtained, in the considered ε-order perturbation of a circular orbit, by simple quadratures of the aerodynamical coefficients.

The autocorrelation functions of each component of the approximated solution process

$$\mathbf{y}^* = \{y_i^*\} = \{\mathbf{u}^*, \mathbf{v}^*\} = \{\mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)}, \mathbf{v}^{(0)} + \varepsilon\mathbf{v}^{(1)}\} \tag{39}$$

are

$$\begin{aligned} &E\{y_i^*(t_1)\, y_i^*(t_2)\} \\ &\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_i^*(t_1)\, y_i^*(t_2)\, P_2(y_i^*(t_1), y_i^*(t_2), t_1, t_2)\, dy_i^*(t_1)\, dy_i^*(t_2) \\ &\quad= y_i^{(0)}(t_1)\, y_i^{(0)}(t_2) + \varepsilon[y_i^{(0)}(t_1)\, E\{y_i^{(1)}(t_2)\} + y_i^{(0)}(t_2)\, E\{y_i^{(1)}(t_1)\}] \\ &\qquad + \varepsilon^2 E\{y_i^{(1)}(t_1)\, y_i^{(1)}(t_2)\}. \end{aligned} \tag{40}$$

In Eq. (40), $y_i^{(0)}$ are supplied by the unperturbed solution (19), (22); $E\{y_i^{(1)}\}$ are easily deduced from the above Eqs. (34)–(38), whereas the autocorrelation functions $E\{y_i^{(1)}(t_1)\, y_i^{(1)}(t_2)\}$ of each component of the perturbed solution can be obtained in terms of the known correlation function matrix $\Gamma_{XX}(t_1, t_2)$ of the stochastic process coefficient $X(t, \omega)$. In particular, by setting $t_1 = t_2 = t$, Eq. (40) yields

$$E\{y_i^{*2}\} = y_i^{(0)2} + 2\varepsilon y_i^{(0)} E\{y_i^{(1)}\} + \varepsilon^2 E\{y_i^{(1)2}\} \tag{41}$$

from which it is obtained

$$\sigma_i^2(t) = \mathrm{Var}\{y_i^*\} = E\{y_i^{*2}\} - E^2\{y_i^*\} = \varepsilon^2\, \mathrm{Var}\{y_i^{(1)}\}. \tag{42}$$

Therefore, the variance of the approximated solution process is of the order of ε^2 and, as a consequence of Eqs. (25), (28), (31), is completely known when the second-order moments of the stochastic process $X(t, \omega)$ are given.

First Probability Density

In order to avoid loss of generality of the results, the analysis so far developed has been carried on without assuming restrictions on the probabilistic model for the air density ρ proposed in the axiom A4. Nevertheless, the analysis of experimental data [6] shows that the air density fluctuations are mainly due to solar effects giving rise to variations on daily or semiannual time-scales. In this connection, if the considered dynamical problem is restricted to an observation time T corresponding to a few satellites' orbits, then a reliable probabilistic model for ρ can be assumed by substituting in

Eq. (5) the stochastic process $X(t, \omega)$ by a deterministic function $\varphi(\boldsymbol{\beta})$ of a given set $\boldsymbol{\beta}$ of random variables

$$\boldsymbol{\beta} = \{\beta_r\} \in D_\beta \subset \mathbb{R}^p \tag{43}$$

joined to suitable constant probability densities $P_\beta(\boldsymbol{\beta})$. This line has been proposed, for instance, in Ref. [18].

It must be outlined that under this hypothesis the first probability density of the solution process can be explicitly calculated by the following procedure. Introduce the augmented state vector

$$\mathbf{z} = \{\mathbf{y}, \boldsymbol{\beta}\}, \qquad \mathbf{z} \in D_z \subset \mathbb{R}^{(12+p)}. \tag{44}$$

The evolution equation $\mathbf{z}$, which is obviously given by Eq. (7) with the addition of the p scalar equations $\dot{\beta}_r = 0$, $r = 1, 2, \ldots, p$, is a differential equaton with *deterministic* parameters and *random* initial conditions

$$\mathbf{z}(t=0) = \mathbf{z}_0(\omega) = \{\mathbf{y}_0, \boldsymbol{\beta}\} \tag{45}$$

which are joined to the known probability density

$$P(\mathbf{z}_0) = \prod_{i=1}^{12} \delta(y_i - y_{i,0}) \, P_\beta(\boldsymbol{\beta}). \tag{46}$$

For such a class of stochastic equations, it is known [8, 10] that the probability density of the solution process $\mathbf{z}$ can be calculated by

$$P(\mathbf{z}, t; \mathbf{z}_0) = P(\mathbf{z}_0) \cdot J(t; \mathbf{z}_0) \tag{47}$$

where J is the Jacobian of the inverse transformation $\mathbf{z}(t) \to \mathbf{z}_0$, satisfying the evolution equation

$$\frac{dJ}{dt} = -J \sum_{i=1}^{12} \frac{\partial}{\partial y_i} \left[g_i(\mathbf{y}) + \varepsilon f_i(\mathbf{y})(1 + \varphi(\boldsymbol{\beta})) \right] \tag{48}$$

where g_i and f_i are defined by Eqs. (9), (10). Calculating in Eq. (48) the right-hand side derivatives and integrating, it is obtained

$$J(t; \mathbf{z}_0) = \exp \left\{ -\int_0^t \left[\frac{\partial}{\partial y_6} g_6(\mathbf{y}(s)) + \varepsilon(1 + \varphi(\boldsymbol{\beta})) \sum_{m=7}^{12} \frac{\partial}{\partial y_m} f_m(\mathbf{y}(s)) \right] ds \right\}. \tag{49}$$

Substitution of Eqs. (46) and (49) into Eq. (47) gives the first probability density at time t for the augmented variable $\mathbf{z}$. Integration of the latter over the domain D_β yields the following first probability density for the state vector $\mathbf{y}(t, \omega)$:

$$P(\mathbf{y}, t; \mathbf{y}_0) = \prod_{i=1}^{12} \delta(y_i - y_{i,0}) \cdot \int_{D_\beta} \exp \left\{ -\int_0^t \left[\frac{\partial}{\partial y_6} g_6(\mathbf{y}(s)) + \varepsilon(1 + \varphi(\boldsymbol{\beta})) \times \sum_{m=7}^{12} \frac{\partial}{\partial y_m} f_m(\mathbf{y}(s)) \right] ds \right\} P_\beta(\boldsymbol{\beta})\, d\boldsymbol{\beta}. \tag{50}$$

If the approximated expression of $\mathbf{y}^* = \{\mathbf{u}^*, \mathbf{v}^*\}$ is used for calculating the integral of Eq. (50), then the last equation supplies values of the first probability density $P(\mathbf{y}^*, t; \mathbf{y}_0)$ within the same order of approximation assumed in the above analysis.

Concluding, in the present work the motion of a spinning sphere in free molecular flow has been studied under the assumption that the aerodynamical forces acting on the body have a random nature, which is specified by means of a probabilistic model of the air mass density. The approximated explicit solution of the dynamical problem for bounded time intervals was obtained in terms of a first-order perturbation expansion of a small deterministic parameter, which in the actual physical conditions of the satellite's flight is of the order of 10^{-9}. The stochastic treatment of the problem was carried on by following the known methods proposed by Adomian [9, 11, 12], Bellomo [10] and Song [8] for studying random evolution equations. The application of these methods supplied analytical results concerning the first- and second-order moments of the solution process and, under suitable conditions to be satisfied by the air density probabilistic model, led to the calculation of the first probability density for the random vector describing the satellite's dynamical state.

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